

NON-ISOTHERMAL ELASTOVISCOPLASTIC ANALYSIS OF PLANAR CURVED BEAMS[†]

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The paper focuses on the development of a general mathematical model and solution methodologies, to examine the behavior of thin structural elements such as beams, rings, and arches, subjected to large non-isothermal elasto-viscoplastic deformations. Thus, geometric as well as material-type nonlinearities of higher order are present in the analysis.

For this purpose a complete true abinitio rate theory of kinematics and kinetics for thin bodies, without any restriction on the magnitude of the transformation is presented. A previously formulated elasto-thermo-viscoplastic material constitutive law is employed in the analysis.

The methodology is demonstrated through three different straight and curved beams problems. Moreover importance of the inclusion of large strains is clearly demonstrated, through the chose applications.

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1. Introduction

The prediction of inelastic behavior of metallic materials at elevated temperatures has increased in importance in recent years. The operating conditions within the hot section of a rocket motor or a modern gas turbine engine present an extremely harsh thermo-mechanical environment. Large thermal transients are induced each time the engine is started or shut down. Additional thermal transients from an elevated ambient occur, whenever the engine power level is adjusted to meet flight requirements. The structural elements employed to construct such hot sections, as well as any engine components located therein, must be capable of withstanding such extreme conditions. Failure of a component would, due to the critical nature of the hot section, lead to an immediate and catastrophic loss in power and thus cannot be tolerated. Consequently, assuring satisfactory long term performance for such components is a major concern for the designer.

Traditionally, this requirement for long term durability has been a more significant concern for gas turbine engines rather than rocket motors. However, with the advent of reusable space vehicles, such as the Space Shuttle, the requirement to accurately predict future performance following repeated elevated temperature operations must now be extended to include the more extreme rocket motor application.

Under this kind of severe loading conditions, the structural behavior is highly nonlinear due to the combined action of geometrical and physical nonlinearities. On one side, finite deformation in a stressed structure introduces nonlinear geometric effects. On the other side, physical nonlinearities arise even in small strain regimes, whereby inelastic phenomena play a particularly important role. From a theoretical standpoint, nonlinear constitutive equations should be applied only in

connection with nonlinear transformation measures (implying both deformation and rotations). However, in almost all of the works in this area (See Ref. 1), the two identified sources of nonlinearities are always separated. This separation yields, at one end of the spectrum, problems of large response, while at the other end, problems of viscous and/or non-isothermal behavior in the presence of small strain.

The classical theories, in which the material response is characterized as a combination of distinct elastic, thermal, time independent inelastic (plastic) and time dependent inelastic (creep) deformation components cannot explain some phenomena, which can be observed in complex thermo-mechanical loading histories. This is particularly true when high-temperature non-isothermal processes should be taken into account. There is a sizeable body of literature^{1,2} on phenomenological constitutive equations for the rate - and temperature - dependent plastic deformation behavior of metallic materials. However, almost all of these new "unified" theories are based on small strain theories and several suffer from some thermodynamic inconsistencies.

In a previous paper³, the authors have presented an alternative constitutive law for elastic-thermo-viscoplastic behavior of metallic materials, in which the main features are: (a) unconstrained strain and deformation kinematics, (b) selection of reference space and configuration for the stress tensor, bearing in mind the rheologies of real materials, (c) an intrinsic relation which satisfies material objectivity, (d) thermodynamic consistency, and (e) proper choice of external and internal thermodynamic variables. Accuracy of the formulation was checked on a wide range of examples⁴.

The formulation presented in this paper focuses on a mathematical model to examine the behavior of thin structural elements subjected to large non-isothermal elasto-viscoplastic deformations. Thus, geometric as well as material-type nonlinearities of higher order are present in the analyses. Such thin elements, including beams, rings and arches, are intended to present generic types of components, which might be located within or adjacent to the hot section of a rocket motor or gas turbine engine.

The rate form of the constitutive equations suggests that⁵ a rate approach be taken toward the entire problem so that flow is viewed as history dependent process rather than an event. A direct consequence of the consistent adoption of the rate viewpoint in a spatial reference frame is that the problem is found to be governed by quasi-linear differential equations in time and in space. Hence, the analysis requires solution of an initial- and boundary- value problem involving instantaneously linear equations. The quasi-linear nature of the problem not only suggests an incremental approach to numerical solution, but also provides confidence in the completeness of the incremental equations. In this case, finite element solution capability is established; it should be noted, however, that the linearity of the instantaneous governing equations admits use of a wide variety of other established numerical procedures for spatial integration. A complete true ab initio rate theory of kinematics and kinetics for continuum and double curved thin structures, without any restriction on the magnitude of the strains or the deformation was formulated in Ref. 4 and will be rephrased here.

Formulation of problems concerned with finite deformation of beams has followed two different paths⁶. Prescribing the beam by its deformed or

undeformed centroidal axis and cross section, one may introduce at the outset beam stress resultants and their conjugate kinematic variables characterizing displacement and rotation of the cross section. Together with appropriate beam constitutive equations and a global balance law a consistent theory is obtained. Alternatively, one may imbed beam theory in the setting of deformable solid continua, in which case one is concerned with local constitutive equations connecting the stress tensor with a strain tensor, which may in turn be expressed in terms of a combination of undetermined beam kinematic variables and functions of the beam coordinates. Momentum may then be balanced globally by integrating the local equations over the deformed beam configuration. Both paths will be considered in what follows.

2. Two-Dimensional Plane Beams (A Plane Stress Problem)

2.2 - Kinematics of the Continuum

Let a continuum in space be described by two systems of coordinates, the x^i -system, which stays at rest (the fixed system) and the u^α -system, which is associated with materials points (the convected material system). The transformation equations from one system to the other are:

$$dx^i = f_\alpha^i du^\alpha \quad (1)$$

$$du^\alpha = u_i^\alpha dx^i \quad (2)$$

where

$$f_\alpha^i \triangleq \frac{\partial x^i}{\partial u^\alpha} \quad (3)$$

$$f_i^\alpha \triangleq \frac{\partial u^\alpha}{\partial x^i} \quad (4)$$

The covariant components of the metric tensor in the material system u^α are:

$$g_{\alpha\beta} = f_\alpha^i f_\beta^j G_{ij} \quad (5)$$

where G_{ij} are the covariant components of the metric tensor in the fixed system x^i . For the fixed cartesian system (Euclidian space) we have,

$$G_{ij} = \delta_{ij} \quad (6)$$

where δ_{ij} are the components of the Kronecker delta. The coordinate lines of the x^i -system are assumed to "deform" with the continuum in order to enable the material points to keep their coordinates (in the u^α -system) unchanged.

The contravariant components of the velocity vector in the fixed system are defined by:

$$v^i = \frac{dx^i}{dt} \quad (7)$$

It is impossible to define velocity as a change in the coordinates in the material system, however, distances are obviously changing. The length of the elementary arc in the material coordinates is given by:

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta \quad (8)$$

Defining the rates of change in the material system by $\frac{\partial}{\partial t}$, the rate of change of the elementary arc is,

$$\frac{\partial}{\partial t}(ds^2) = \frac{\partial g_{\alpha\beta}}{\partial t} du^\alpha du^\beta \quad (9)$$

From Eq (9) one may conclude also that

$$\frac{1}{ds} \left(\frac{\partial(ds)}{\partial t} \right) = \frac{\partial}{\partial t} (\log ds) = \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial t} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \quad (10)$$

The clue for the intrinsic rates of change may be unraveled, then, by the derivation of the rates of change of the metric tensor.

It can be shown⁴ that the rate of change of the metric tensor in the material coordinates is given by

$$\frac{\partial g_{\alpha\beta}}{\partial t} = v_{\alpha,\beta} + v_{\beta,\alpha} \quad (11)$$

where

$$v_{\alpha,\beta} = g_{\gamma\alpha} v^{\gamma}_{,\beta} = g_{\gamma\alpha} f^{\gamma}_i v^i_{,\beta} = g_{\gamma\alpha} f^{\gamma}_i \frac{\partial}{\partial t} (f^i_{\beta}) \quad (12)$$

So $\frac{\partial}{\partial t}(f^i_{\beta})$ or $v^i_{,\beta}$ is the velocity in the fixed system as observed in the material system. The components of the deformation rate tensor defined as follows,

$$d_{\alpha\beta} \triangleq \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial t} = \frac{1}{2} (v_{\alpha,\beta} + v_{\beta,\alpha}) = d_{\beta\alpha} \quad (13)$$

and the components of the spin tensor as,

$$\omega_{\alpha\beta} \triangleq \frac{1}{2} (v_{\alpha,\beta} - v_{\beta,\alpha}) = -\omega_{\beta\alpha} \quad (14)$$

Substitution of Eq. (13) into Eq. (10) yields,

$$\frac{\partial}{\partial t} (\log ds) = d_{\alpha\beta} \lambda^{\alpha} \lambda^{\beta} \quad (\lambda^{\alpha} = \frac{du^{\alpha}}{ds}) \quad (15)$$

As soon as the deformation rate is established as the time derivative of the metric tensor, the intrinsic characteristics of the continuum, being metric properties of space, are readily differentiated (with respect to time). For more details see Ref. 4.

2.2 The Rate of Global Principles

The principle of virtual power (or of virtual velocities),

$$\int_V \sigma^{ij} \delta v_{j,i} dV - \int_V \rho f^j \delta v_j dV - \int_A v T^j \delta v_j dA = 0 \quad (16)$$

is equivalent to the equations of equilibrium along with the complete set

of boundary conditions. In Eq. (16), σ^{ij} are the contravariant components of the Cauchy stress tensor, ρ the mass density, and f^j is a vector of specific body forces.

Total differentiation of Eq. (16) yields⁴,

$$\begin{aligned} & \int_V \left(\frac{d\sigma^{ij}}{dt} + \sigma^{ij} d_{.k}^k = v^i_{,k} \sigma^{kj} \right) \delta v_{j,i} dV - \int_V \rho \frac{df^j}{dt} \delta v_j dV \\ & - \int_A \gamma \frac{dT^j}{dt} \delta v_j dA + \int_V \sigma^{ij} \frac{d\delta v_j}{dt} , i dV - \int_V \rho f^j \frac{d\delta v_j}{dt} dV \\ & - \int_A \gamma T^j \frac{d\delta v_j}{dt} dA = 0 \end{aligned} \quad (17)$$

At any instant Eq. (17) must be satisfied. The virtual velocity and its time derivative are, then, independent. Moreover, the last three terms of Eq. (17) are equivalent to Eq. (16). Hence, the principle of the rate of virtual power may be obtained in its concise form. For further classifications, the total derivative of the stress components will be represented by the Jauman derivative, namely

$$\frac{d\sigma^{ij}}{dt} = \nabla_i \sigma^{ij} + \omega^i_{.k} \sigma^{kj} + \omega^j_{.k} \sigma^{ik} \quad (18)$$

and the following integrals are defined by

$$I_e = \int_V \nabla_i \sigma^{ij} \delta v_{j,i} dV \quad (19)$$

$$I_d = \int_V \left(\sigma^{ij} d_{.k}^k - \sigma^{kj} d_{.k}^i \right) \delta v_{j,i} dV \quad (20)$$

$$I_r = \int_V \omega^j_{.k} \sigma^{ik} \delta v_{j,i} dV \quad (21)$$

Then, substitution in Eq. (17) yields the final form of the principle of the rate of virtual power,

$$I = I_e + I_d + I_r = \int_r \rho \frac{df^j}{dt} \delta v_j dV + \int_A \gamma \frac{dT^j}{dt} \delta v_j dA \quad (22)$$

which is equivalent to

$$\frac{d\sigma^{ij}}{dt},_i - v^k_{,i} \sigma^{ij}_{,k} + \rho \frac{df^j}{dt} - \rho d^k_{,k} f^j = 0 \quad (23)$$

and

$$\frac{d(vT^j)}{dt} = \frac{d(\sigma^{ij}v_j)}{dt} \quad (24)$$

A similar process can be applied⁴ to obtain the principle of the rate of balance of energy from the first law of thermodynamics.

2.3 Constitutive Equations

In a previous paper³, the authors have presented a complete set of constitutive relations for nonisothermal, large strain, elasto-viscoplastic behavior of metals. It was shown there³ that the metric tensor in the convected (material) coordinate system can be linearly decomposed into elastic and (visco) plastic parts. So a yield function was assumed, which is dependent on the rate of change of stress on the metric, on the temperature and a set of internal variables. Moreover, a hypoelastic law was chosen to describe the thermo-elastic part of the deformation.

A time and temperature dependent "viscoplasticity" model was formulated in this convected material system to account for finite strains and rotations. The history and temperature dependence were incorporated through the introduction of internal variables. The choice of these variables, as well as their evolution, was motivated by thermodynamic considerations.

The nonisothermal elasto-viscoplastic deformation process was described completely by "thermodynamic state" equations. Most investigators^{1,2} (in the area of viscoplasticity) employ plastic strains as state variables. This study³ shows that, in general, use of plastic strains as state variables may lead to inconsistencies with regard to thermodynamic considerations. Furthermore, the approach and formulation employed in previous works leads to the condition that all the plastic work is completely dissipated. This, however, is in contradiction with experimental evidence, from which it emerges that part of the plastic work is used for producing residual stresses in the lattice, which, when phenomenologically considered, causes hardening. Both limitations were excluded from this formulation.

The constitutive relation will be rephrased here as follows

$$a) \text{ if } F = (t_k^i - C \rho_0 g \beta_k^i)(t_i^k - C \rho_0 g \beta_i^k) - k^2 \overset{P}{W, T} = 0 \quad (25)$$

where, s_k^i , the Kirchhoff stress tensor, $s_k^i = \frac{\rho_0}{\rho} \sigma_k^i$, and the temperature T are independent process variables, t_k^i being the deviator of the Kirchhoff

stress. s_k^i , and $\overset{P}{W}$ and β_k^i are internal parameters, then

$$\dot{d}_k^i = \underbrace{\frac{1}{2Gv} \left\{ s_k^i - \frac{v}{1+v} s_r^r \delta_k^i \right\} + \alpha \dot{T} \delta_k^i}_{E_{d_k^i}} + \underbrace{2\dot{\lambda} (\bar{t}_k^i - C \rho_0 g \beta_k^i)}_{P_{d_k^i}} \quad (26)$$

with

$$\dot{\lambda} = \frac{1}{4n} \left\{ \left[\frac{(t_k^i - C \rho_0 g \beta_k^i)(t_i^k - C \rho_0 g \beta_i^k)}{k^2} \right]^{1/2} - 1 \right\} \quad (27)$$

$$\bar{t}_k^i = \frac{1}{1 + 4 n \dot{\lambda}} (t_k^i - c \rho_0 g \beta_k^i) + c \rho_0 g \beta_k^i \quad (28)$$

$$\dot{W} = \frac{1}{\rho_0} \bar{t}_k^i d_i^k \quad (29)$$

$$\nabla_{\beta_k^i}^i = \zeta d_k^i \quad (30)$$

$$b) \quad \text{if} \quad F = 0 \quad (31)$$

$$\text{and} \quad \frac{\partial F}{\partial s_k^i} s_k^i + \frac{\partial F}{\partial T} \dot{T} > 0 \quad (32)$$

$$\text{then} \quad d_k^i = d_k^i \quad (33)$$

$$d_k^i = 0 \quad \nabla_{d_k^i}^i = 2 \ddot{\lambda} (t_k^i - c \rho_0 g \beta_k^i) \quad (34)$$

$$\text{with} \quad \ddot{\lambda} = \frac{1}{8 n k^2} \{ 2 (t_k^i - c \rho_0 g \beta_k^i) \nabla_{t_i^k}^k - \frac{\partial k^2}{\partial T} \dot{T} \} \quad (35)$$

$$c) \quad \text{if} \quad F = 0 \quad \text{and} \quad \frac{\partial F}{\partial s_k^i} s_k^i + \frac{\partial F}{\partial T} \dot{T} \leq 0 \quad (36)$$

$$\text{or} \quad F < 0 \quad (37)$$

$$\text{then} \quad d_k^i = \frac{E_i}{d_k^i} \quad (38)$$

$$\begin{matrix} P \\ \cdot \\ W = 0 \end{matrix} \quad (39)$$

$$\nabla \beta_k^i = 0 \quad (40)$$

2.4 Plane Stress Approximation

By definition, a body is said to be in the state of plane stress parallel to the u^1, u^2 plane when the stress components $\sigma^{13}, \sigma^{23}, \sigma^{33}$ vanish⁷. It is well known in literature that the case of plane stress is difficult to handle theoretically. Even linear elasticity has to treat this case in an approximate manner. To remove some of theoretical difficulties Durban and Baruch⁵ introduced the notion of Generalized Plane Stress, where instead of dealing with the quantities themselves, one deals with their average values.

In our case the problem is even more difficult. The nonlinearities, which the general three-dimensional theory takes into account will also cause a large change of the geometrical quantities in the u^3 direction. Clearly, some assumptions are needed to treat the case of plane stress as a two-dimensional case.

The first basic assumption is that the thickness, h , of the plate defined by the coordinates u^1, u^2 located in its middle plane, is small as compared with the other two dimensions. A second assumption is that the

external forces act in the u^1 , u^2 directions and are symmetrically distributed with respect to the middle plane.

In a way similar to the procedure proposed by Durban and Baruch⁵, all the kinematic expressions are obtained by averaging the three-dimensional expressions.

A basic assumption for the case of plane stress is that the components connected with the third direction are small and can be neglected. So, a new concept of generalized stress tensor is introduced

$$\tau_k^i = \frac{\sigma_k^i h}{h_0} \quad (41)$$

It must be noted that in the linear theory of elasticity, where the geometry does not change, the averaged and generalized stress tensors coincide.

So the three-dimensional incremental elasto-viscoplastic theory, developed previously, can be adopted for two-dimension plane stress problems.

3. A Thin Curved Beam

3.1 Doubly Curved Element

A complete rate theory of kinematics and kinetics for doubly and singly curved thin structures, without any restriction on the magnitude of the strain or the deformation, was presented in Ref. 4.

Five different shell theories (approximations), in rate form, starting with the simple Kirchhoff-Love theory and finishing with a completely unrestricted one, were considered there⁴.

The kinematic and kinetic equations for intrinsic shell dynamics, introduced in Ref. 4 are presented here, in compact form, together with basic notations. For simplicity we consider here Kirchhoff motion only⁸.

The components of the velocity vector $w = \dot{y}$ are

$$w_\alpha = \dot{y} \cdot y_\alpha \quad w^\alpha = \dot{y} \cdot y^\alpha, \quad w_n = \dot{y} \cdot \bar{n} \quad (42)$$

where \bar{n} is the unit normal to y , and y is the weighted motion.

Expressions for the components of the velocity gradients, \dot{y} , follow from differentiation of Eq. (42)

$$d_{\alpha\beta} = y_\alpha \cdot \dot{y}_\beta = \omega_{\alpha;\beta} - \bar{b}_{\alpha\beta} w_n \quad (43)$$

$$\tilde{w}_\alpha = \bar{n} \cdot \dot{y}_\beta = \omega_{n,\alpha} + \bar{b}_{\alpha\beta} w^\beta \quad (44)$$

The time rates of the components of the metric and curvature tensors follow immediately from the above as

$$\dot{\bar{a}} = d_{\alpha\beta} + d_{\beta\alpha}, \quad \dot{\bar{b}}_{\alpha\beta} = \tilde{\omega}_{\alpha;\beta} + d_\alpha^\gamma \bar{b}_{\beta\gamma} \quad (45)$$

To complete the kinematics, we get the components of the acceleration vector by time differentiation of Eq. (42) and through use of Eqs. (43) and (44),

$$\ddot{y} \cdot y^\alpha = \dot{w}^\alpha + d_\beta^\alpha w^\beta - \tilde{\omega}^\alpha w_n \quad (46)$$

$$\ddot{y} \cdot \bar{n} = \dot{w}_n + \tilde{\omega} w^\alpha$$

The accelerations form the right sides of the equations of motion. The left sides are the static terms that can, for example, be expressed in terms of symmetrical stress resultants⁴. The result becomes

$$\dot{w}^\alpha = \bar{m}^{-1} [\bar{n}^{\beta\alpha}_{;\beta} + \bar{b}^\alpha_{\lambda;\beta} + \bar{m}^{\lambda\beta} + 2\bar{b}^\alpha_\lambda \bar{m}^{\beta\lambda}_{;\beta} + \bar{p}^\alpha] - d_\beta^\alpha w^\beta + \tilde{\omega}^\alpha w_n \quad (47)$$

$$\dot{w}_n = \bar{m}^{-1} [\bar{n}^{\alpha\beta} \bar{b}_{\alpha\beta} + \bar{b}_{\beta}^{\lambda} \bar{b}_{\alpha\lambda} \bar{m}^{\alpha\beta} - \bar{m}^{\alpha\beta}_{;\alpha\beta} + \bar{p}] - \bar{\omega}^{\alpha} w_{\alpha}. \quad (48)$$

Here \bar{p}^{α} , \bar{p} , \bar{m} are loading components and mass, respectively, per unit area of y .

3.2 A Simplified Version of Curved Beam Element

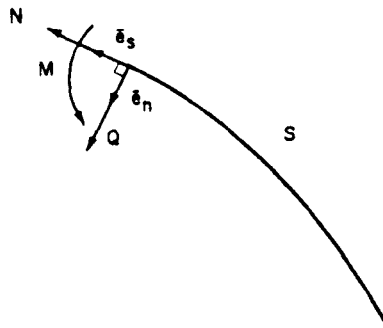


Fig. 1 - Reference Line of a Curved Beam

A portion of the reference line for a curved beam is shown on Fig. 1. The current arc length is denoted by s , while ϕ is the current angle of inclination of the normal to the reference line, and ρ is the radius of curvature.

The stress resultants acting on the beam cross section are the bending moment M , the axial force N , and the shear force Q . The external load, measured per unit of current length of the reference line, has the components p_s and p_n in the direction of the unit vectors \bar{e}_s and \bar{e}_n respectively.

If v_s and v_n denote the velocity components in the direction of the unit vectors \bar{e}_s and \bar{e}_n respectively, the rate of extension is

$$d = \frac{\partial v_s}{\partial s} - \frac{v_n}{\rho} \quad (49)$$

The rate of rotation, $\dot{\phi}$, of a given section is given by

$$\omega = \dot{\phi} = \frac{\partial v_n}{\partial s} + \frac{v_s}{\rho} \quad (50)$$

while the generalized rate of deformation associated with bending is

$$\dot{k} = \frac{\partial \dot{\phi}}{\partial s} = -\frac{\partial}{\partial s} \left(\frac{\partial v_n}{\partial s} + \frac{v_s}{\rho} \right) \quad (51)$$

The rate of equilibrium equations for this simplified version may be put in the following form:

$$\begin{aligned} \frac{\partial \dot{N}}{\partial s} - Q \frac{\partial \phi}{\partial s} - \dot{Q} \frac{\partial \phi}{\partial s} + \dot{p}_s + d p_s &= 0 \\ \frac{\partial \dot{Q}}{\partial s} + N \frac{\partial \phi}{\partial s} + \dot{N} \frac{\partial \phi}{\partial s} + \dot{p}_n + d p_n &= 0 \\ \frac{\partial \dot{M}}{\partial s} + \dot{Q} + d Q &= 0 \end{aligned} \quad (52)$$

4. Numerical Solution

The quasi-linear nature of the velocity equilibrium equations suggests the adoption of an incremental approach to numerical integration with respect to time. The availability of the field formulation provides assurance of the completeness of the incremental equations and allows the use of any convenient procedure for spatial integration over the domain B . In the present instance the choice has been made in favor of a simple first order expansion in time for the construction of incremental solutions from

the results of finite element spatial integration of the governing equations.

The procedure employed permits the rates of the field formulation to be interpreted as increments in the numerical solution. This is particularly convenient for the construction of incremental boundary condition histories.

The finite element method for spatial discretization has been well documented (see, e.g. Zienkiewicz⁹ or Oden¹⁰) and will not be detailed here. It should be noted, however, that as a consequence of the present formulation, the velocity equilibrium equations are not symmetric. This feature precludes implementation of a Ritz procedure as commonly employed in finite element analysis of infinitesimal deformation. Linear algebraic equations governing the discrete model for the finite case are developed employing the method of Galerkin.

The spatial discretization results in:

$$[K] \{v\} = \{\dot{R}\} \quad (53)$$

where $[K]$ is the nonsymmetric stiffness matrix, $\{v\}$ is the vector containing the generalized nodal velocities, and $\{\dot{R}\}$ is the rate of the load. The solution to Eq. (53) at time t provides a basis for evaluation of a deformation increment and associated changes in internal stresses and boundary loading. The incremental solution defines the deformed configuration and stress rate at $t = t_0 + \delta t$ thereby permitting definition of a new spatial problem at the later time.

5. Applications

The capabilities of the models presented here in have been evaluated through three simple numerical examples. The first example demonstrates the capability of the plane stress approximation to predict deflections and

processes in a beam loaded by a constant moment. Figure 2 illustrates the beam and the finite element model. A quarter of the beam was divided into six elements in the vertical direction and into five elements in the horizontal direction. The external moment was introduced by six parallel forces acting on the section BC (see Fig. 2).

The value of the external moment is 3500 kg/cm, and the material of the beam is CHR-17. The viscoplastic properties of the material were obtained experimentally from uniaxial tests in Ref. 1. This properties were collaborated into the present material model.

The variation of the deflection of point E as a function of time is given in Fig. 3. It is important to point out the value of the large deformation analysis. After ten minutes of the deformation is increased by 43% and at the same time there are important changes in the stress field (see Fig. 4).

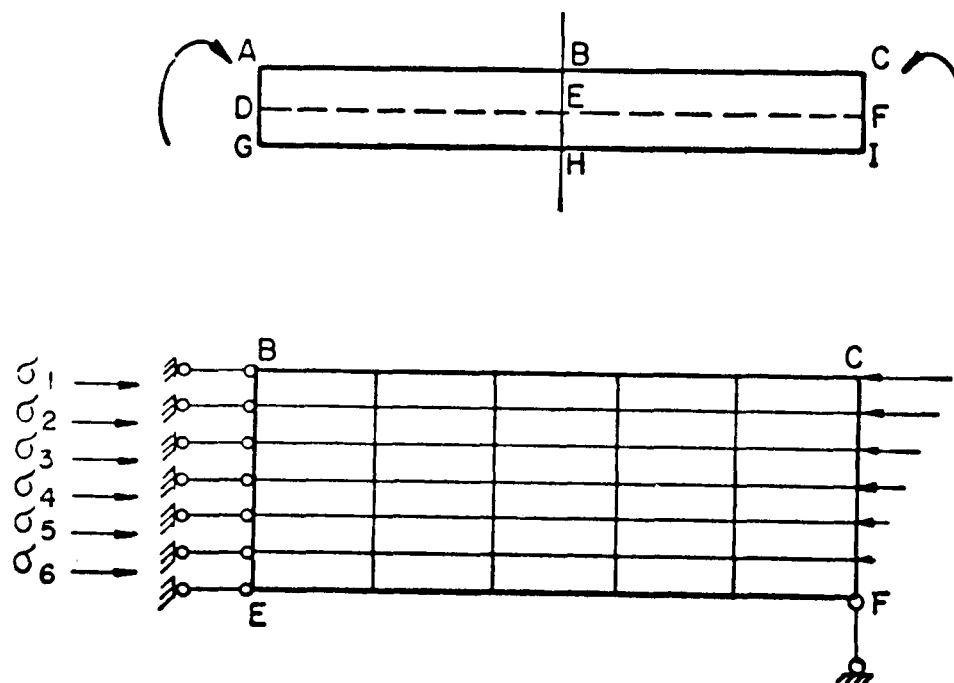


Fig. 2 - The Beam Model

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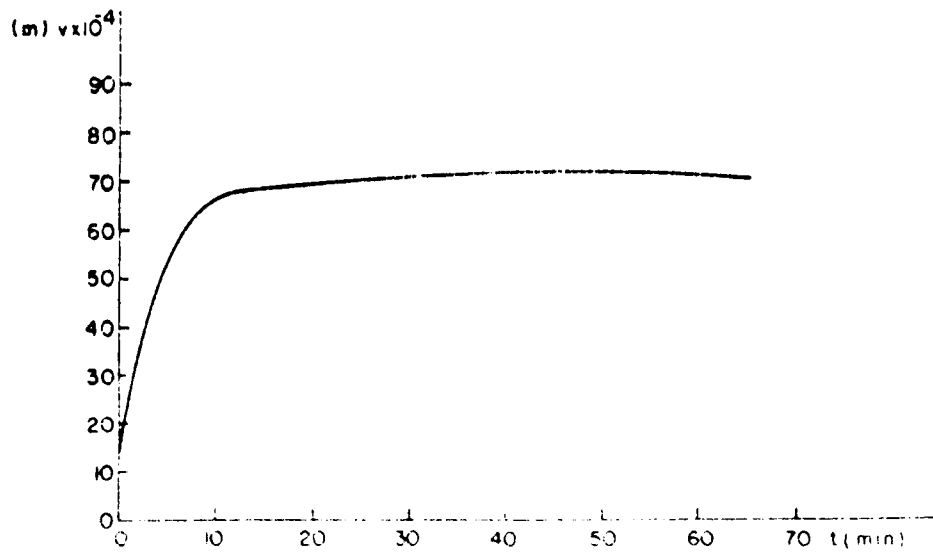


Fig. 3 - Point E Deflection

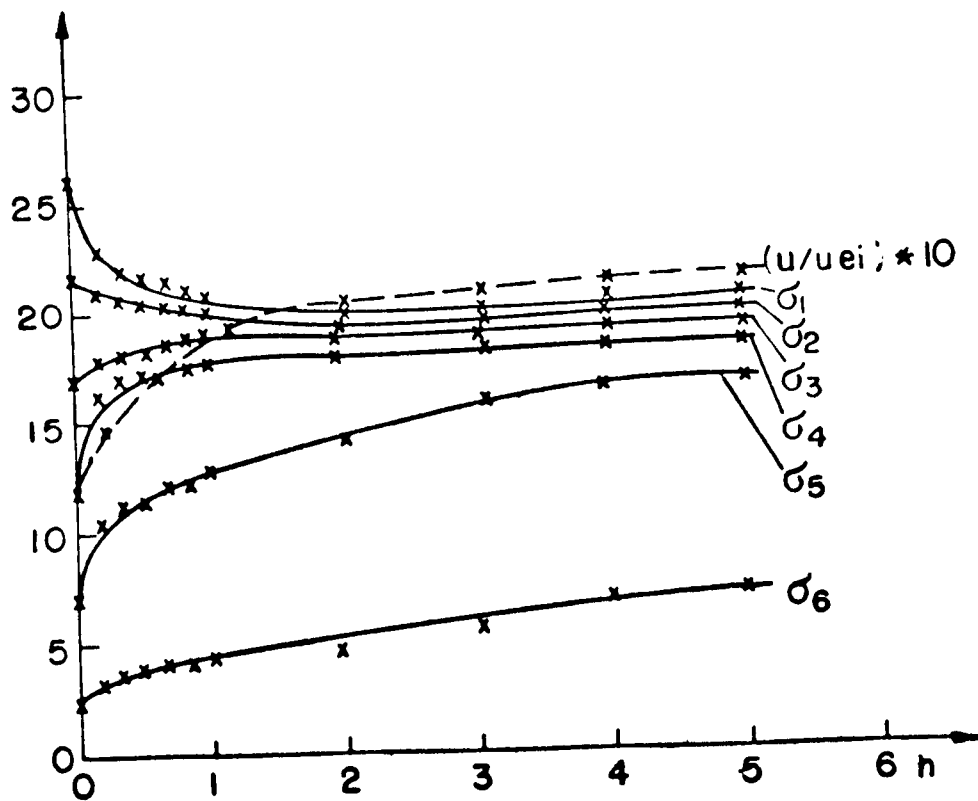


Fig. 4 - Stress Distribution

The next example consists of a straight simply supported beam, loaded by a transverse concentrate force at the midspan. The beam is 25 inches long, two inches high and one inch wide. The material is stainless steel 304 (Heat 9T2796). The material constants in sub section 2.3 were correlated with the uniaxial tension experimental results given in Ref. 12. The beam was subjected to a load of 2000 pounds at 1100°F, this load was then held constant for 312 hr., and then increased to 2250 pounds at 1400°F.

The primary purpose of this example is to compare the results, obtained by the two previously discussed models. The first one is the two-dimensional plane stress model, and the second one is the thin beam model as derived from thin shell theory. Figure 5 presents results in the form of load versus midspan deflection. The finite element model consists of five simple plane stress elements (dashed line in Fig. 5) or five sophisticated beam elements (full line in Fig. 5).

It can be seen (Fig. 5) that the results agree quite well up to the 312-hour hold period (points 3,4). During the hold period, the material hardens and only the beam model can represent this behavior after the load is further increased.

The last example presents an analysis of a circular arch. The geometry of the shallow circular arch is shown on Fig. 6. The material is once again the 304 stainless steel. The arch is fixed at both ends and carries a concentrated load at the center. The elasto-viscoplastic analysis of this arch is performed with the aid of a ten curved beam element model and with the inertia terms taken into account. The load P is assumed to be applied in a quasi-static manner at $t = 0$. The results of this analysis are shown on Fig. 6, as the time-history of the midspan

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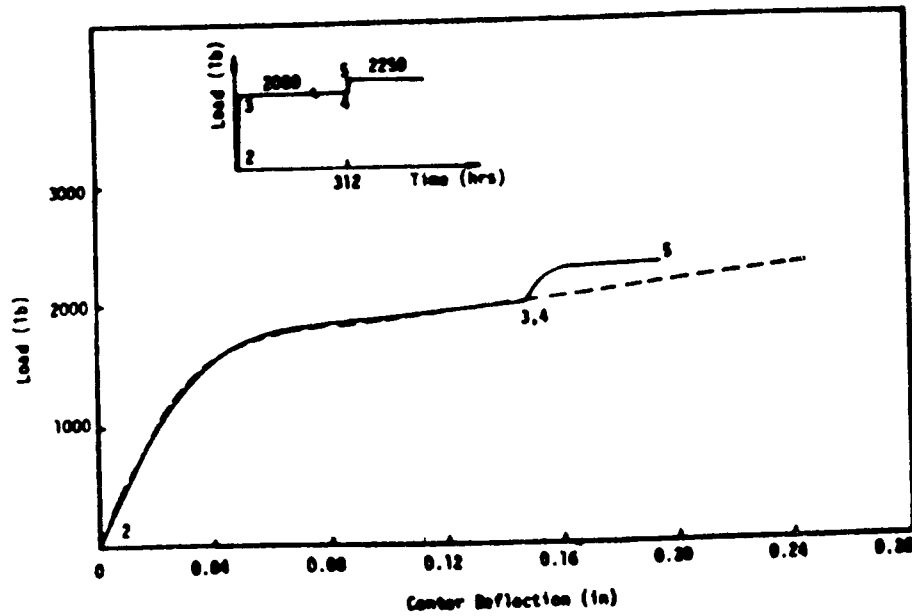


Fig. 5 - Deflection vs Load

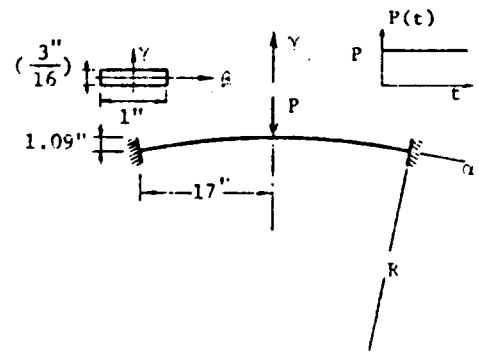
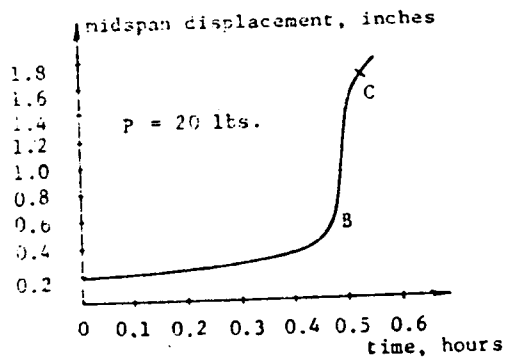


Fig. 6 - Circular Shallow Arch

displacement. The response of the arch starts with the instantaneous elastic deformation at $t = 0$, followed by slow deformation up to point B, which can be considered as a limit point for the given value of the load P . Beyond point B, the displacements increase rapidly towards point C. This may suggest the existence of critical time for the prescribed load.

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